MIXED DISPLACEMENT-PRESSURE STABILIZED FINITE ELEMENT FORMULATION FOR GEOMECHANICS

ELEMENTS FINIS STABILISES EN FOMULATION MIXTE « DEPLACEMENT-PRESSION » APPLIQUES A LA GEOMECANIQUE

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ABSTRACT - A mixed stabilized finite element formulation is derived in this paper for elastoplasticity, with applications in geomechanics. A modified Galerkin least-squares (GLS) formulation is developed and shown to overcome volumetric locking phenomena in both incompressible and dilatant plasticity, even for low-order elements. The performance of the modified GLS formulation is demonstrated on two benchmarks.

RESUME - Une formulation mixte et stabilisée d’éléments finis pour l’élastoplasticité est proposée dans cet article, avec application à la géomécanique. On montre qu’une formulation Galerkin aux moindres carrés (GLS) modifiée empêche le verrouillage volumétrique dans les cas de plasticité incompressible et dilatante, et ce même pour des éléments d’ordre inférieur. Cette formulation est validée sur deux exemples.

1. Introduction

Stabilized finite element formulations have received much attention in recent years, mainly in the fluid mechanics domain (Brooks and Hughes, 1982; Hughes et al., 1989)), with the main objective of improving solutions in convection dominated flows when using low-order elements.

In the context of soil and rock mechanics applications, our aim is to overcome locking phenomena associated with incompressible elasticity and incompressible or dilatant plasticity. Spurious oscillations in the pressure field should be eliminated, and the simultaneous use of different low-order elements - like the linear triangle and the bilinear quad - in the same mesh is also a desirable feature.

A stabilized Galerkin least-squares formulation (Hughes et al, 1986) is modified and applied to a mixed displacement-pressure finite element formulation of elastoplasticity. Analogies with finite increment calculus (Oñate, 2000) and Laplacian pressure operator stabilization (Pastor et al, 1997) are briefly pointed out (for more details, see Commend and Zimmermann, 2002b).

The performance of the stabilized formulation is demonstrated on an academic example, namely the thick cylinder test loaded by an internal pressure, as well as on a typical soil mechanics problem, the bearing capacity of a superficial footing. Results demonstrate that the mixed displacement-pressure finite element formulation with stabilization provides an appropriate solution to these problems in both the incompressible and the dilatant case, supports element mixture of low-order quads and triangles within the same mesh and therefore has significant advantages over previously used approaches (BBAR, enhanced-assumed strains, higher-order elements, cross-diagonal triangular pattern).

2. Mixed formulation of elastoplasticity

Governing equations for the mixed displacement-pressure formulation of elastoplasticity are given in Table 1. $\sigma = [\sigma_{11}, \sigma_{22}, \sigma_{12}, \sigma_{33}, \sigma_{13}, \sigma_{23}]^T$ is the three-dimensional stress vector. $\mathbf{D}$ is the deviatoric projection of the elastic constitutive matrix, and $\mathbf{e}$ is the total strain vector, decomposed into an elastic part $\mathbf{e}^e$ and a plastic part $\mathbf{e}^p$. $\mathbf{u}$ is the displacement field, and $p$ the pressure field.
\[ \mathbf{I} = [1, 1, 0, 1, 0, 0]^T \text{ is Kronecker's vector. } \Delta \gamma \text{ is the plastic multiplier, } f \text{ the yield function and } g \text{ the plastic potential. } \Delta \text{ stands for an incremental value.} \]

In Equations (6)-(9) we consider a body \( \Omega \), with boundary \( \partial \Omega = \Gamma \cup \Gamma_h \), with normal \( n \). \( f_i \) are body forces and \( K \) is the bulk modulus. A comma denotes differentiation.

**Table I. Governing equations for elastoplasticity (mixed formulation)**

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>( \Delta \sigma = \mathbf{D}[\Delta \epsilon(u) - \Delta \epsilon^p(u, p)] + \mathbf{1} \Delta p )</td>
<td>(1) Incremental constitutive equation</td>
</tr>
<tr>
<td>( \Delta \epsilon(u, p) = \Delta \epsilon^e(u) - \Delta \epsilon^p(u, p) )</td>
<td>(2) Additive decomposition of the strain field</td>
</tr>
<tr>
<td>( f(\sigma) = 0 )</td>
<td>(3) Yield function</td>
</tr>
<tr>
<td>( \Delta \gamma \hat{f}(\sigma) = 0 )</td>
<td>(4) Flow rule</td>
</tr>
<tr>
<td>( L^T \sigma + f = 0 \text{ or } \sigma_{h,j} + f_j = 0 )</td>
<td>(6) Equilibrium in ( \Omega )</td>
</tr>
<tr>
<td>( u_j' - \frac{p}{K} = 0 )</td>
<td>(7) Volumetric constitutive equation in ( \Omega )</td>
</tr>
<tr>
<td>( u_i = g_i )</td>
<td>(8) Dirichlet boundary condition on ( \Gamma_i )</td>
</tr>
<tr>
<td>( \sigma_{e,n_j} = h_i )</td>
<td>(9) Neumann boundary condition on ( \Gamma_n )</td>
</tr>
</tbody>
</table>

### 3. Stabilization

#### 3.1 The modified Galerkin least-squares (GLS) formulation

We obtain a GLS formulation by adding a least-squares term integrated over elements (Equation (11)) to the second line of the standard Galerkin form (Equation (10)) (see Hughes et al, 1989).

\[
\int_{\Omega} \mathbf{e}(\mathbf{w}^h)^T \mathbf{\sigma}(\mathbf{u}^h, p^h) d\Omega = \int_{\Gamma_h} \mathbf{h}^T d\Gamma + \int_{\Omega} \mathbf{w}^h^T f d\Omega \\
\int_{\Omega} q^h \mathbf{1}^T \mathbf{e}^e(\mathbf{u}^h) d\Omega - \int_{\Omega} q^h \frac{p^h}{K} d\Omega = 0 \\
\sum_{e=1}^{n_e} \int_{\Omega} \mathbf{1}^T \mathbf{L}^T \mathbf{\sigma}(\mathbf{w}^h, q^h)^T \mathbf{\tau}(\mathbf{L}^T \mathbf{\sigma}(\mathbf{u}^h, p^h) + f) d\Omega \\
\]

(11)

\( \mathbf{L} \) is the differential operator defined in Equation (6) and \( \mathbf{\tau} \) a stabilization factor. A straightforward application of the procedure leads to weighting terms which include plastic components (see Equation (12)), precluding a straightforward implementation. The plastic part can, however, be neglected, leading to a modified GLS formulation (see Commend et al, 2002a).

\[
\mathbf{L}^T \mathbf{\sigma}(\mathbf{w}^h, q^h) = \mathbf{L}^T \left[ \mathbf{D}[\Delta \epsilon(w^h) - \Delta \epsilon^p(w^h, q^h)] + \mathbf{1} q^h \right] \\
\]

(12)

Equation (13) provides a definition for stabilization factor \( \mathbf{\tau} \), by analogy with Hughes et al, 1986.

\[
\mathbf{\tau} = \mathbf{\tau} \cdot \mathbf{I} = \frac{\alpha}{2\mu} \mathbf{I} \\
\]

(13)
3.2 Other stabilization schemes

Commend, 2001 shows that formulations with pressure stabilization only (i.e. neglecting also the total strain increment part in Equation (12)) lead to simpler schemes. A similar and equally successful scheme based on the Laplacian of the pressure (LPOS) was proposed by Pastor et al, 1997, and additional ones can be derived, for solids, along the lines advocated by Oñate, 2000 for fluid flow, as demonstrated in Commend and Zimmermann, 2002b.

4. Examples

4.1 The thick cylinder test

A thick cylinder loaded by an internal pressure is analysed first (see Figure 1). The original analytical solution (Hill, 1950) involves Tresca’s yield criterion \( \sigma_0 - \sigma_r = \sigma_y \). In the incompressible plane strain case, an equivalent von Mises criterion can be expressed with \( k = \sigma_y / \sqrt{3} \).

![Figure 1. Thick cylinder test geometry and parameters](image1)

E = 21000 kN/m²
\( \nu = 0.49999 \)
\( k = \sigma_y / \sqrt{3} = 13.8564 \) kN/m²

\( a = 1.0 \) m
\( b = 2.0 \) m
\( c \) denotes the plastic radius
\( p = 8 \rightarrow 20 \) kN/m (total plastification)

![Figure 2. Circumferential stress distribution across the cylinder for p = 14 kN/m](image2)
A mesh composed of 160 quadrilateral elements is considered first. The theoretical solution for the circumferential stress, taken from Hill, 1950, is compared in Figure 2 with the solution obtained with stabilized finite elements (modified GLS formulation for different values of the $\alpha^e$ parameter) and with the BBAR formulation. Large oscillations appear in the unstabilized mixed formulation ($\alpha^e = 0.0$), which also exhibits a strong volumetric effect (see Commend et al, 2002a). Oscillations are still noticeable for $\alpha^e = 0.01$. The optimal value for $\alpha^e$ is located in the lower part of the interval [0.1, 1.0], with higher values of $\alpha^e$ the circumferential peak stress cannot be retrieved anymore. Notice also the poor approximation of stresses at the internal radius, which is a boundary effect due to a coarse discretization.

Mixing triangles and quads in an unstructured mesh reveals that the solution involving BBAR quadrilateral and standard triangular elements violates looses axisymmetry for the last converged step (Figure 3a), while the stabilized solution ($\alpha^e = 1.0$ for both quads and triangles) restores axisymmetry (Figure 3b).

![Figure 3](image)

**Figure 3.** Displacement intensities for $p = 19$ kN/m (last converged step)

### 4.2 The bearing capacity of a superficial footing

A superficial foundation (Figure 4) is loaded by a uniform pressure increased until a failure mechanism is detected. An analytical solution to this problem has been published by Matar and Salençon, 1979.

![Figure 4](image)

**Figure 4.** Superficial footing geometry and characteristics

When standard bilinear displacement quadrilateral elements are used, results show an overshoot of theoretical solutions (Figure 5), characteristic of locking. This also holds for constant strain triangles, or when a mixture of the two types of elements is being used. Earlier solutions to overcome locking, like strain-projection (BBAR), enhanced assumed strains EAS or cross-
diagonal triangles all solve the incompressible case for quads; EAS also overcomes locking in dilatant media when the mesh is composed of quads, but all fail when quads and triangles are used simultaneously within the same mesh. The proposed stabilized formulation is shown, in Figure 5, to provide a distinct ultimate load in the incompressible case, with a mesh composed of quads, triangles, or both (see Figure 6). The soil is incompressible in this example, but Commend, 2001, shows that the same formulation also performs well in the dilatant case. $\alpha^e$ (Equation (13)) can be arbitrarily selected in the range 0.1 to 1.0.

Figure 5. Evolution of the vertical displacement under the footing

Figure 6. Mesh and failure mechanism
5. Conclusion

A mixed-displacement pressure stabilized formulation is proposed and shown to appropriately fight numerical instabilities and volumetric locking on two examples (additional examples are given in Commend and Zimmermann, 2000c or Commend, 2001). The advantages of such stabilized schemes on other formulations are: their applicability to any type of element (unlike BBAR or enhanced assumed strains which do not apply to linear triangles), their validity in both the incompressible and the dilatant plastic case, and the fact that they allow to mix different low-order elements in the same mesh.

The LPOS formulation (Pastor et al, 1997) is slightly cheaper, from the numerical performance point of view, and gives results comparable to the modified GLS scheme.

A larger system of equations is required when compared to standard finite elements, due to the mixed formulation.

6. Acknowledgements

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7. References